

# Laplace of Some other functions:

	$F(t)$	$L(F(t))=f(s)$
<b>Sine Integral,</b>	$S_i(t)=\int_0^t \frac{\sin u}{u} du$	$\frac{1}{s} \tan^{-1}\left(\frac{1}{s}\right)$
<b>Cosine Integral,</b>	$C_i(t)=\int_t^\infty \frac{\cos u}{u} du$	$\frac{\log(s^2 + 1)}{2s}$
<b>Exponential Integral,</b>	$E_i(t)=\int_t^\infty \frac{\cos u}{u} du$	$\frac{\log(s + 1)}{s}$
<b>Error function,</b>	$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2} du$	$\frac{1}{s\sqrt{s+1}}$
<b>Comp. Error function,</b>	$\operatorname{erfc}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^\infty e^{-u^2} du$	$\frac{1}{s} - \frac{1}{s\sqrt{s+1}}$
<b>Bessel function of zero order* ,</b>	$J_0(at)$	$\frac{1}{\sqrt{s^2 + a^2}}$
	$\operatorname{Sin}\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$
<b>Periodic Function,</b>	$F(t + nT) = F(t) ,$ T>0 being Period and n∈set of Integers	$\frac{\int_0^T e^{-su} F(u) du}{1 - e^{-sT}}$
<b>Convolution:</b>	$F(t)*G(t)=\int_0^t F(u) G(t-u) du$	$f(s).g(s)$

\* Bessel function of n<sup>th</sup> order is

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)} \left[ 1 - \frac{1}{2 \cdot (2n+2)} t^2 + \frac{1}{2 \cdot 4 (2n+2)(2n+4)} t^4 - \dots \right]$$

# Laplace Transformations(Contd.)

**Sufficient condition** for existence of Laplace Transform

There are two governing factors that determine whether Laplace transforms of a function can be used:

- $F(t)$  must be at least piecewise continuous for  $t \geq 0$
- $|F(t)| \leq Me^{\gamma t}$  where  $M$  and  $\gamma$  are constants.(i.e. $F(t)$  is of **exponential order**  $\gamma$ .)

Above *conditions are sufficient but not necessary* for the existence of Laplace Transform . If these conditions are not satisfied, however, Laplace transform may or may not exist.

For Example  $L(\frac{1}{\sqrt{t}}) = \sqrt{\frac{\pi}{s}}$  , but it does not satisfy sufficient condition for the existence of Laplace.

Another example :  $e^{t^3}$  is not of exponential order since  $|e^{-\gamma t}e^{t^3}| = e^{t^3-\gamma t}$  can be made larger than any given constant by increasing  $t$ .

## Other Applications of Laplace Transformations:

1. **Evaluations of Integrals:** we know  $L\{F(t)\} = f(s)$  or  $\int_0^\infty e^{-st} F(t) dt = f(s)$ , thus  $f(a)$  can be calculated for any particular value of  $s = a$ . For example,  

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

2. **In solving Integral equations of Convolution type:** For example  

$$Y(t) = t^2 + \int_0^t Y(u) \sin(t-u) du$$

After Laplace transforms it becomes  $y = \frac{2}{s^3} + L\{Y * \sin t\} = \frac{2}{s^3} + \frac{y}{s^2+1}$

Simplifying,  $y = \frac{2}{s^3} + \frac{2}{s^5}$

by Inversion,  $Y = t^2 + \frac{1}{12} t^4$ . It can be verified directly from integral equations.

3. **Solving Bending Beam problems:** The differential equation for transverse deflection  $Y(x)$  from one end  $x = 0$  at a distance  $x$  for a uniform beam is

governed by 
$$\frac{d^4 Y}{dx^4} = \frac{W(x)}{EI}; 0 < x < l.$$
  $W(x)$  being *vertical load per unit*

*length*.  $EI$  is called *flexural rigidity* of the beam which is assumed constant. The B.c's of differential equation depend on the manner in which beam is supported

as: (i) *Clamped, Built-In or Fixed End:*  $Y = Y' = 0$ ; (ii) *Hinged, Simply supported End:*  $Y = Y'' = 0$ ;

(iii) *Free End*:  $Y''' = Y'' = 0$ .

## Legendre's Polynomial:

The series solution of the o.d.e.  $(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$

.....(A) is called a **Legendre's Polynomial  $P_n(x)$**  of degree  **$n$**  and defined by

$$P_n(x) = \frac{1.3.5 \dots (2n-1)}{n!} \left[ x^n - \frac{n(n-1)}{2 \cdot (2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - \dots \right]$$

$$\text{Or } P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n r! (n-r)! (n-2r)!} x^{n-2r}$$

$$\text{Where } \left[ \frac{n}{2} \right] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

**Generating Function :**  $(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$  ;  $|x| \leq 1, |h| \leq 1$ ,  
**Rodrigue's formula:**  $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$

## Legendre's Polynomial(Contd.)

**Orthogonal properties:**

$$\int_{-1}^1 P_m(x) P_n(x) dx = 0 \text{ and } \frac{2}{2n+1} \text{ according as } m \neq n \text{ and } m=n$$

**Christoffel's expansion:**

$P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \dots + 3P_1$  or  $P_0$  as  $n$  is even or odd.

**Recurrence relations:**

$$(i) n P_n = (2n-1)xP_{n-1} - (n-1)P_{n-2},$$

$$(ii) n P_n = xP'_n - P'_{n-1},$$

$$(iii) (2n+1)P_n = P'_{n+1} - P'_{n-1}$$

$$(iv) (n+1)P_n = P'_{n+1} - xP'_n$$

$$(v) (1-x^2)P'_n = nP_{n-1} - nxP_n$$

$$(vi) (1-x^2)P'_n = (n+1)(xP_n - P_{n+1})$$

## Legendre's Polynomial(Contd.)

### Properties:

$$(i) P_n(1) = 1,$$

$$(ii) P_n(-1) = (-1)^n,$$

$$(iii) P'_n(1) = \frac{1}{2} n(n+1)$$

$$(iv) P'_n(-1) = \frac{1}{2} (-1)^{n-1} \cdot n(n+1)$$

(v)  $P_n(-x) = (-1)^n P_n(x)$  and hence we can explain the nature of  $P_n(x)$  when  $n$  is even or odd.

$$(vi) P_{2m+1}(0) = 0, \text{ and } P_{2m}(0) = (-1)^m \frac{2m!}{(m!)^2 2^{2m}}$$

### Applications:

1. Integrals  $\int_{-1}^1 x P_n(x) P_{n-1}(x) dx = \frac{2n}{4n^2-1}$  ;and

$$\int_{-1}^1 (1-x^2) P'_m(x) P'_n(x) dx = 0, \text{ } m \neq n \text{ can be evaluated.}$$

2.  $x^4 + 2x^3 + 2x^2 - x - 3$  can be expressed in terms of Legendre's polynomial.

$$\frac{8}{35} P_4(x) + \frac{4}{5} P_3(x) + \frac{40}{21} P_2(x) + \frac{1}{5} P_1(x) - \frac{224}{105} P_0(x)$$

3.  $P_n(x) = \frac{1}{\pi} \int_0^\pi [x \pm \cos \phi \sqrt{x^2 - 1}]^n d\phi$ , if  $n$  is a positive integer.

## FOURIER SERIES:

Let  $F(x)$  satisfies the following sufficient conditions (*Dirichlet conditions*):

- i.  $F(x)$  is defined in the interval  $c < x < c + 2l$ .
- ii.  $F(x)$  and  $F'(x)$  have finite number of discontinuities in the interval  $c < x < c + 2l$ .
- iii.  $F(x)$  is periodic with period  $2l$ .

Then *at each point of continuity*, we have

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \left( \frac{n\pi x}{l} \right) + b_n \sin \left( \frac{n\pi x}{l} \right) \right\} \quad \text{.. F1}$$

Where,  $a_n = \frac{1}{l} \int_c^{c+2l} F(x) \cos \left( \frac{n\pi x}{l} \right) dx$

$$b_n = \frac{1}{l} \int_c^{c+2l} F(x) \sin \left( \frac{n\pi x}{l} \right) dx$$

## FOURIER SERIES (Contd.)

At a **point x of discontinuity**,  $F(x)$  must be replaced by mean of two limits  $F(x - 0)$  and  $F(x + 0)$  i.e.  $\frac{F(x-0) + F(x+0)}{2}$  in the L.H.S of F1 in the previous Slide.

*The following integrals are very useful in determination of Fourier Coefficients:*

$\forall m, n \in \mathbb{Z}$ .

$$(i) \int_c^{c+2l} \cos \left( \frac{m\pi x}{l} \right) \sin \left( \frac{n\pi x}{l} \right) dx = 0,$$

$$(ii) \int_c^{c+2l} \cos \left( \frac{m\pi x}{l} \right) \cos \left( \frac{n\pi x}{l} \right) dx = 0, \quad \text{if } m \neq n$$



$$= \pi, \quad \text{if } m = n (\neq 0),$$

***Important:***

$$(iii) \int_c^{c+2l} \sin\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = 0.$$

All above results are also true within the limits  $-\pi$  to  $\pi$ ,  $0$  to  $2\pi$  etc.(which can be obtained using  $c = -\pi$  and  $l = \pi$ ;  $c = 0$  and  $l = 2\pi$  respectively in the above results. )

## FOURIER SERIES (Contd.)

**Helpful facts to evaluate Fourier Coefficients :**

(1) All integrals discussed above vanish except the integral

$$\int_c^{c+2l} \cos^2\left(\frac{m\pi x}{l}\right) dx = \pi. \quad \text{if } m \neq 0.$$

$$\sin n\pi = 0, \quad \cos n\pi = (-1)^n, \quad \forall n \in \mathbb{Z}.$$

(2) Sometimes, we have to use the property of definite integrals:

$$\int_{-c}^c f(x) \, dx = 2 \int_0^c f(x) \, dx, \text{ if } f(-x) =$$

$f(x)$  i. e.  $f(x)$  is an even function.

$$\int_{-c}^c f(x) \, dx = 0, \text{ if } f(-x) = -f(x) \quad \text{i.e. } f(x) \text{ is an odd function.}$$

**$x^2$ ,  $\cos x$ ,  $\sin^2 x$  etc. are examples of even functions\***  
**and  $x^3$ ,  $\sin^3 x$ ,  $x \cos x$  etc. are examples of odd functions.\*\***

\*we can check graphically, even functions are always *symmetric about y-axis*.

\*\* odd functions are always *symmetric in opposite quadrants*.

## **FOURIER SERIES (Contd.)**

### **PERIODIC FUNCTIONS:**

A function  $f(x)$  is said to be periodic function if  $f(x + nT) = f(x)$ ,  $n \in \mathbb{Z}$ , where  $T$  being the period.

For example,  $\sin x$ ,  $\cos x$ ,  $\sec x$  and  $\csc x$  are periodic functions with period  $2\pi$ .

$\tan x$  and  $\cot x$  are periodic functions with period  $\pi$ .

**\* Sum of two periodic functions is again periodic** but its period will change, which can be evaluated under the following rules:

- (1) If period of  $f(x)$  is  $T$ , then period of  $f(nx)$  is  $T/n$ .
- (2) If period of  $f_1(x)$  is  $T_1$  and  $f_2(x)$  is  $T_2$ , then period of  $a f_1(x) + b f_2(x)$  is *L.C.M. of  $T_1$  and  $T_2$* .

**FOURIER SERIES:** Let  $F(x)$  satisfy at least the following **sufficient** conditions:

- $F(x)$  is defined in the interval  $c < x < c + 2l$ .
- $F(x)$  and  $F'(x)$  have finite number of discontinuities in the interval  $c < x < c + 2l$ .
- $F(x)$  is periodic with period  $2l$ .

These conditions are known as *Dirichlet conditions*.

Then at each point of continuity, we have

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \left( \frac{n\pi x}{l} \right) + b_n \sin \left( \frac{n\pi x}{l} \right) \right\},$$

$$\text{where } a_n = \frac{1}{l} \int_c^{c+2l} F(x) \cos \left( \frac{n\pi x}{l} \right) dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} F(x) \sin \left( \frac{n\pi x}{l} \right) dx$$

At a point  $x$  of discontinuity,  $F(x)$  must be replaced by mean of two limits  $F(x-0)$  and  $F(x+0)$  i.e.  $\frac{F(x-0) + F(x+0)}{2}$  in the L.H.S of above.

The following integrals are very useful in determination of Fourier Coefficients:

$\forall m, n \in \mathbb{Z}$ .

$$(i) \quad \int_c^{c+2l} \cos \left( \frac{m\pi x}{l} \right) \sin \left( \frac{n\pi x}{l} \right) dx = 0,$$

$$(ii) \quad \int_c^{c+2l} \cos \left( \frac{m\pi x}{l} \right) \cos \left( \frac{n\pi x}{l} \right) dx = 0, \quad \text{if } m \neq n$$

$$= \pi, \quad \text{if } m = n (\neq 0), \quad \dots \text{ Remember}$$

$$(iii) \quad \int_c^{c+2l} \sin \left( \frac{m\pi x}{l} \right) \sin \left( \frac{n\pi x}{l} \right) dx = 0.$$

All above results are also true within the limits  $-\pi$  to  $\pi$  or  $0$  to  $2\pi$  etc. (which can be obtained using  $c = -\pi$  and  $l = \pi$ ;  $c = 0$  and  $l = 2\pi$  respectively in the above results.)

We are giving the derivation of one of these results. Others can be derived in the same manner.

**Proof:** for  $\int_c^{c+2l} \cos \left( \frac{m\pi x}{l} \right) \sin \left( \frac{n\pi x}{l} \right) dx$

$$= \frac{1}{2} \int_c^{c+2l} \left\{ \sin(m+n)\left(\frac{\pi x}{l}\right) - \sin(m-n)\left(\frac{\pi x}{l}\right) \right\} dx$$

{Since  $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ }

$$= -\frac{1}{2} \left[ \frac{\cos(m+n)\left(\frac{\pi x}{l}\right)}{m+n} - \frac{\cos(m-n)\left(\frac{\pi x}{l}\right)}{m-n} \right]_c^{c+2l}, \quad m \neq n$$

$$= 0, \quad m \neq n$$

**Remark:**

- Reader can observe yourself that **all the integrals discussed above vanish except the integral**  $\int_c^{c+2l} \cos^2 \left( \frac{m\pi x}{l} \right) dx = \pi$ . **if  $m \neq 0$ .**

$$\sin n\pi = 0, \cos n\pi = (-1)^n, \forall n \in \mathbb{Z}.$$

- Sometimes, we have to use the property of definite integrals:

$$\int_{-c}^c f(x) dx = 2 \int_0^c f(x) dx, \text{ if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is an even function.}$$

$$\int_{-c}^c f(x) dx = 0, \text{ if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is an odd function.}$$

$x^2, \cos x, \sin^2 x$  etc. are examples of even functions\*

and  $x^3, \sin^3 x, x \cos x$  etc. are examples of odd functions.\*\*

\*we can check graphically, even functions are always *symmetric about y-axis*.

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- We know that the integration of the product of two functions can be solved using generalized rule of integration by parts (According to this rule choice of first and second function can be decided by the order of the letters in **ILATE**, **I** stands for **inverse**

functions such as  $\sin^{-1} x$ ,  $\cos^{-1} x$  etc; **L** stands for **logarithmic** functions such as  $\log x$ ,  $\log(\sin x^2)$  etc; **A** stands for **Algebraic** functions such as  $x^2$ ,  $3x+x^2$  etc.; **T** stands for **Trigonometric** functions such as  $\sin x$ ,  $\cos x$  etc.;

**E** stands for **Exponential** functions such as  $e^x$ ,  $e^{\sin x}$  etc. But, to evaluate the integration of the product of two functions in which **one of them must be a polynomial function** it would be better to use the following **time saving technique**.

$$\int PQ dx = PQ_1 - P^1 Q_2 + P^2 Q_3 - \dots \dots \dots$$

Here, above rule is applicable if at *least one of P and Q* is a polynomial function of  $x$ . The superscript denotes the differentiation and subscript denote the integration. e. g.  $P^2$  Denotes second derivative of  $P$  w.r. t.  $x$  and  $Q_3$  denotes third times integration of  $Q$  w.r. t.  $x$ . Since  $P$  is a **polynomial function** so its derivative becomes zero after a finite number of steps.

Applications:

$$(1). \int x^2 \sin x \, dx = x^2(-\cos x) - 2x(-\sin x) + 2(\cos x).$$

$$(2). \int x^3 e^{2x} \, dx = \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{6}{8} x e^{2x} - \frac{6}{16} e^{2x}.$$

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For example,

- I.  $\sin 2x$ ,  $\cos 3x$ ,  $\tan 4x$  are periodic functions with period  $\frac{2\pi}{2}$ ,  $\frac{2\pi}{3}$ ,  $\frac{\pi}{4}$  respectively. (follow rule 1)
- II.  $\sin 2x + \frac{1}{4} \cos 3x + \frac{1}{3} \tan 4x$  is also a periodic function with period **L.C.M. of  $\frac{2\pi}{2}$ ,  $\frac{2\pi}{3}$ ,  $\frac{\pi}{4}$**  i.e.  $2\pi$ . (from rule 2)